Tutorial

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Robustness of Complex Networks

8th International Workshop on the Design of Reliable Communication Networks

Krakow, Poland, October 10-12, 2011
Robustness of Complex Networks

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Network Architecture and Services (NAS)

Outline

Introduction
Graph metrics
Spectrum
Network models
Attacks & failures
Framework for robustness
Birth of graph theory: the Königsberg bridge problem (Euler, 1736)

Can one walk across the seven bridges and never traverse the same bridge twice?

Eulerian walk: zero or two nodes with odd degree

What is a network?

A graph $G(N, L)$ specifies how items, called nodes, are interconnected or related to other nodes by links.

- Trees: $L = N - 1$
- Ring: $L = N$
- Complete graph: $L = N(N-1)/2$
**Fractal Nature of the Internet**

The average human cerebral cortex contains

\[ N = 10^{11} \] neurons

\[ L = 10^{14} \] connections

500,000 km of wiring

*moon-earth: 384 405 km*
Network Science

• Are there properties common to all complex networks?
• if so, why?
• Can we formulate a general theory of the structure (topology), evolution and dynamics of complex networks?
• How do complex networks give rise to “adaptive”, “living”, “intelligent” behavior?
• How can we learn from nature to design robust, efficient, self-adaptive “man-made” networks?

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**graph metric: degree**

degree $d_j$ of node $j$: number of neighbors of $j$

- $N = 6$
- $L = 9$

- $d_1 = 3$
- $d_2 = 4$
- $d_3 = 3$
- $d_4 = 2$
- $d_5 = 3$
- $d_6 = 3$

$$\sum_{j=1}^{N} d_j = 2L$$

average degree in G equals

$$E[D] = \frac{1}{N} \sum_{j=1}^{N} d_j = \frac{2L}{N}$$

bounds: $2 - \frac{2}{N} \leq E[D] \leq N - 1$

---

**Adjacency matrix $A$**

For an undirected graph: $A = A^T$ is symmetric

Number of neighbors of node $i$ is the degree: $d_i = \sum_{k=1}^{N} a_{ik}$
**Adjacency matrix** $A$

**Degree vector** $d$: $A u = d$ and $u = (1,1,...,1)$

**Walk of length** $k$ **from node** $i$ **to** $j$: succession of $k$ links (arcs)  
$(n_0 \rightarrow n_1)(n_1 \rightarrow n_2)\ldots(n_{k-1} \rightarrow n_k)$ where $n_0 = i$ and $n_k = j$

**Path**: a walk in which all nodes/vertices are different

**Number of** $k$-**hop walks** between node $i$ and $j$: $(A^k)_{ij}$

**Open**: **Number of** $k$-**hop paths** between node $i$ and $j$ in terms of adjacency elements

---

**Graph metric: degree**

Airline transportation network

![Airline network graph](image)

Internet: $\Pr[D_{\text{Internet}} = k] \sim k^{-\gamma}$, $\gamma \in (2.2,2.5)$

![Degree distribution graph](image)

$$\Pr[D_{\text{Air}} = k] \sim k^{-1.21}$$
**Graph Metric: Clustering coefficient**

The clustering coefficient of node $v$ is

$$c_G(v) = \frac{2y}{d_v(d_v - 1)}$$

where $y$ is the number of links between neighbors. If $d_v = 1$, $c_G(v) = 0$.

$$c_G(A) = \frac{2}{4(4-1)} = \frac{1}{6}, \quad c_G(C) = 1$$

The clustering coefficient of a graph $G$:

$$c_G = \frac{1}{N} \sum_{v \in V} c_G(v)$$

**Graph Metric: Hopcount**

hopcount $H$: number of links in a shortest path in $G$

$$N = 6 \quad L = 9$$

$$H_{14} = 2$$

diameter of $G$: hopcount of the longest shortest path in $G$

average hopcount $E[H]$ reflects "efficiency" of transport in $G$
**Graph Metric: Betweenness**

The betweenness \( B_{ij} \) of a link \( l \) is the number of shortest paths between all possible node pairs in \( G \) that traverse the link.

\[
H_G = \sum_{i=1}^{N} \sum_{j=i+1}^{N} H_{i-j} = \sum_{i=1}^{L} B_i \quad \text{and} \quad E[B] = \left( \frac{N}{2} \right) E[H_N] \geq E[H_N]
\]

\( H_{i-j} \) is the hopcount of the shortest path between \( i \) and \( j \).

**Assortativity**

How are \( D_i \) and \( D_j \) (cor)related?

\[
\rho_D = \frac{E[D_i D_j] - E[D_i] E[D_j]}{\sqrt{Var[D_i]} \sqrt{Var[D_j]}} \quad \text{link} \ l \quad \rho_D = \frac{E[D_i D_j] - E[D_i] E[D_j]}{\sqrt{Var[D_i]} \sqrt{Var[D_j]}} \quad \text{link} \ l
\]

A network is (degree) **assortative** if \( \rho_D > 0 \)

A network is (degree) **disassortative** if \( \rho_D < 0 \)
(dis)assortativity

Reformulation of Newman’s definition into algebraic graph theory

\[
\rho_D = \frac{E[D_i D_i] - E[D_i^2]}{\sqrt{\text{Var}[D_i] \text{Var}[D_i]}} = \frac{N_1 N_3 - N_2^2}{N_1 \sum_{j=1}^{N} d_j^3 - N_2^2}
\]

where \( N_k = u^T A^k u \) is the total number of walks with \( k \) hops:

\[
N_0 = \sum_{j=1}^{N} d_j^0 = N \quad N_1 = \sum_{j=1}^{N} d_j^1 = 2L \quad N_2 = \sum_{j=1}^{N} d_j^2 = \sigma^T \sigma
\]

\[
N_k \leq \sum_{j=1}^{N} d_j^k
\]


Degree-preserving rewiring

Only two terms change

degree-preserving rewiring algorithm

How many graphs do there exist with given \( N \) and \( \sigma^T \sigma \)?

Open question (see B. McKay)
Connectivity of a Graph

- **Edge connectivity**
- **Vertex connectivity**

\( \lambda(G) \) (or \( \kappa(G) \)) : the minimum number of links (nodes) whose removal disconnects \( G \)

**Menger’s Theorem**:
The maximum number of link(node)-disjoint paths between A and B is equal to the minimum number of links(nodes) separating A and B.

Important inequality: \( \kappa(G) \leq \lambda(G) \leq d_{\min}(G) \leq \frac{2L}{N} \)

There are at least \( \lambda(G) \) link-disjoint and at least \( \kappa(G) \) node-disjoint paths between any pair of nodes in \( G \)
List of topological metrics (undirected, unweighted graphs)

- hopcount
- closeness
- eccentricity
- diameter
- radius
- girth
- expansion
- distortion
- degree
- entropy
- joint degree

- assortativity
- modularity
- coreness
- clique number
- clustering coefficient
- rich club coefficient
- size giant component
- (node/link) connectivity
- coloring
- effective graph resistance
- *and many more*

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Eigenvalues and eigenvectors

\[ A x = \lambda x \]

\[ A \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \]

\[ A X = X \Lambda \quad \Rightarrow \quad A = X \Lambda X^{-1} \]

\[ A = A^T = X \Lambda X^T = \sum_{k=1}^{N} \lambda_k x_k x_k^T \]

Basic theorem for symmetric matrices

Any real symmetric matrix \( S \) can be written as \( S = X \Lambda X^T \), where \( X \) is the orthogonal matrix with real eigenvectors in the columns and \( \Lambda = diag(\lambda_1, \ldots, \lambda_N) \), where \( \lambda_j \) is the \( j \)-th real eigenvalue.

The real eigenvalues can be ordered as

\[ \lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 \]

The eigenvalues are the zeros of the characteristic polynomial

\[ \det(A - \lambda I) = 0 \]
Algebraic graph theory

Any graph $G$ with $N$ nodes and $L$ links can be represented by an adjacency matrix $A$ and an incidence matrix $B$, and a Laplacian $Q$

\[
Q = BB^T = \Delta - A
\]

\[
\Delta = \text{diag}(d_1, d_2, \ldots, d_N)
\]

Basic properties of graph spectra

Spectrum of $A$: 1) all eigenvalues lie in the interval $(-d_{\text{max}}, d_{\text{max}})$

\[
\sum_{j=1}^{N} \lambda_j = 0 \quad \sum_{j=1}^{N} \lambda_j^2 = 2L \quad \sum_{j=1}^{N} \lambda_j = \text{Trace}(A^2) = \sum_{j=1}^{N} (A^2)_{jj}
\]

2) Perron-Frobenius Theorem: $\lambda_1$ non-negative and components eigenvector are non-negative. (irreducible = connected: positive)

Spectrum of $Q$: 1) any eigenvalue $\mu_k$ is non-negative and the smallest $\mu_N = 0$

2) complexity (number of spanning trees) is $\xi(G) = \frac{1}{N!} \prod_{i} \mu_i$

3) the second smallest eigenvalue, 

\[\text{algebraic connectivity} \ a(G) = \mu_{N-1}\]

is related to how strongly a graph is connected

There exists a wealth of properties of graph spectra: see e.g.

P. Van Mieghem, Graph Spectra of Complex Networks, Cambridge University Press, 2011
Refreshing your knowledge (1/2)

• Adjacency matrix

\[ A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} \]

• \( \lambda_1 \) = spectral radius = largest eigenvalue of \( A \)

Refreshing your knowledge (2/2)

• Laplacian matrix

\[ Q = \Delta - A \]

\[ \Delta = \text{diag}(d_1, d_2, \ldots, d_n) \]

\[ Q = \begin{pmatrix}
2 & -1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 1 \\
\end{pmatrix} \]

• \( a(G) \) = algebraic connectivity = second smallest eigenvalue of \( Q \)
Connectivity of graphs

\[
\begin{align*}
G_1 & \quad a(G_1) = 0 \\
G_2 & \quad a(G_2) = 0.29 \\
G_3 & \quad a(G_3) = 0.59
\end{align*}
\]

Largest eigenvalue of a symmetric matrix

If \( Ax = \lambda x \) then \( A^k x = \lambda^k x \) for nonnegative integers \( k \)

Power method: \( A^k w = \alpha_i \lambda_i^k x_i \left( 1 + O \left( \left\| \frac{\lambda_2}{\lambda_1} \right\|^k \right) \right) \)

Gershgorin’s theorem: \( \lambda_1 \leq d_{\text{max}} \)

Rayleigh principle: \( \lambda_1 \geq \frac{w^T A w}{w^T w} \) with equality only if \( w = x_1 \)

There are many variations possible on the Rayleigh principle:
1) find suitable vector \( w \)
2) apply to powers of \( A \) recalling that \( N_k = u^T A^k u \) is the total number of walks with \( k \) hops

P. Van Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, 2011
Bounds largest eigenvalue adjacency matrix

Classical bounds: \[ d_{\text{max}} \geq \lambda_1(A) \geq \frac{2L}{N} = E[D] \]

Walks based: \[ \lambda_1(A) \geq \left( \frac{N_{2k}}{N} \right)^{1/(2k)} \geq \left( \frac{N_k}{N} \right)^{1/k} \]

Optimized: \[ \lambda_4(A) \geq \frac{NN_3 - N_1N_2 + \sqrt{(NN_3)^2 - 6NN_1N_2N_3}}{2} + \text{others} \]

Optimization

- Remove \( m \) nodes in \( G \) such that each removal decreases \( \lambda_1(A) \) maximally.
- Remove \( l \) links in \( G \) such that each link removal decreases \( \lambda_3(A) \) maximally.
  - What are the optimal strategies?
- Unfortunately, these problems are NP-complete...

The Interlacing Theorem

For a real symmetric \( n \times n \) matrix \( A \) and any principal \( m \times m \) submatrix \( B \) of \( A \) obtained by deleting \( n-m \) same rows and columns in \( A \), the eigenvalues of \( B \) interlace with those of \( A \) as \[ \lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A) \text{ for any } 1 \leq i \leq m \]
USA air transportation network
N = 2179 and L = 31326


Degree-preserving rewiring USA air transport network: adjacency eig.
Degree-preserving rewiring USA air transport network: Laplacian eig.

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Attacks & failures

Framework for robustness
Network Model: Erdös-Rényi random graph

It is a class of graphs with $N$ nodes and each node pair is connected independently with probability $p$.

$$E[L] = \frac{N(N-1)}{2} p$$

The average clustering coefficient follows

$$E[c_{G_{p}(N)}] = p$$
Network Model: Erdös-Rényi random graph

Degree distribution: Binomial distribution

\[ \Pr[D_{rg} = k] = \binom{N-1}{k} p^k (1 - p)^{N-1-k} \]

For large \( N \) and \( p = \lambda/N \) approaches a Poisson distribution

\[ \Pr[D_{rg} = k] = \frac{(pN)^k}{k!} e^{-\lambda p} \]

Simplest proof via pgf:

\[ E[z^{D_{rg}}] = (1 - p + pz)^{N-1} = \left(1 + \frac{\lambda(z-1)}{N}\right)^{N-1} \rightarrow e^{\lambda(z-1)} \]

The critical link density \( p_c \)

\[ \Pr[G_p(N) \text{ is connected}] = \Pr[D_{\min} \geq 1] \]

\[ \Pr[G_p(N) \text{ is connected}] = \begin{cases} 
0 & \text{if } p < \log N \\ 
1 & \text{if } p > \log N 
\end{cases} \]

\[ \Pr[G_p(N) \text{ is connected}] = \exp\left(-Ne^{-E_z[0]}\right) \quad p_c \sim \log N/N \]
Random Graph $G_{0.002}(300)$

- Connected cluster size = 12 nodes
- $E[D] = 0.6$, $p = p_c/4$

Random Graph $G_{0.004}(300)$

- Connected cluster size = 25 nodes
- $E[D] = 1.2$, $p = 0.5 p_c$
Random Graph $G_{0.008}(300)$

- Connected cluster size = 255 nodes
- Critical threshold: $p_c \approx 0.019$
- $E[D] = 2.4$

Random Graph $G_{0.016}(300)$

- Connected graph, $p = 2p_c$
- $E[D] = 4.8$
Random Graph $G_{0.032}(300)$

Connected graph, $p = 4p_c$

$E[D] = 9.6$

Eigenvalues $x$

Isolated node
Node with degree $\leq 9$
Node with degree $> 9$

Wigner’s semicircle law for $G_p(N)$

$E[\lambda_1] = p(N-2) + 1$

P. Van Mieghem, Graph Spectra of Complex Networks, Cambridge University Press, 2011
Network Model: Small-world graph

Collective dynamics of ‘small-world’ networks

Duncan J. Watts & Steven H. Strogatz
Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

Table 1 Empirical examples of small-world networks

<table>
<thead>
<tr>
<th>Network</th>
<th>$d$</th>
<th>$d_{random}$</th>
<th>$C_{actual}$</th>
<th>$C_{random}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Film actors</td>
<td>3.65</td>
<td>2.99</td>
<td>0.78</td>
<td>0.00027</td>
</tr>
<tr>
<td>Power grid</td>
<td>15.2</td>
<td>12.4</td>
<td>0.060</td>
<td>0.005</td>
</tr>
<tr>
<td>C. elegans</td>
<td>2.65</td>
<td>2.25</td>
<td>0.28</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Network Model: Small-world graph
Network Model: BA power law graph

Metabolic network: nodes(chemicals) and links(bio-chemical reactions)

**Power laws** in complex networks

![CCDF](image)

### Power law graphs

Measurements of the degree of nodes in (subgraphs of) the Internet topology indicate that

\[
\Pr[D = k] = ck^{-\beta}
\]

A power law degree distribution is also called scale-free:

\[
\Pr[D = ak] = ca^{-\tau}k^{-\beta} = a^{-\tau} \Pr[D = k]
\]

Any number \(a\) just multiplies the probability density; there is no characteristic length.

Moreover,

\[
E[D] = \sum_{r=1}^{\tau-1} \frac{\zeta(r)}{\zeta(r)} \frac{\zeta(r-1)}{\zeta(r)} \text{ provided } r > 2, \text{ where } \zeta(r) = 1 + \gamma + O(r-1)
\]

\[
E[D^+] = \sum_{r=1}^{\tau-1} \frac{\zeta(r-k)}{\zeta(r)} \text{ provided } r > k + 1
\]

The simplest family of "power law" graphs have been proposed by Barabasi-Albert:

1) start with \(n\) nodes
2) attach a new node with \(m\) links to a node proportionally to its degree
3) repeat 2) until size \(N\) is reached

This construction of "preferential attachment", "rich get richer", is observed in many large complex networks (webgraph, proteins, social relations, etc...)
Mystery of “power laws”

Power law of a “property” appear if the “system” grows exponentially:

- if $X$ grows exponentially with $Y$ and $Y$ has an exponential distribution, then $X$ will have a power-law distribution (Proof PA, p. 324)

The exponential function $f(t)$ has a linear differential equation

$$\frac{df(t)}{dt} = a f(t)$$

which essentially means "growing proportional to its size"

At phase transitions, quantities of interest also change in a “power law” fashion
Observed common properties

- *small-world property*
  - average length of a path is short compared to the size $N$ of the network ($E[H] = O(\log N)$)
- *scale-free degree distribution*
  - heavy tails (non-Gaussian behavior)
- *clustering and community structure*
  - network of networks
- *robustness to random node failure*
- *vulnerability to targeted hub attacks and cascading failures*

Outline

- Introduction
- Graph metrics
- Spectrum
- Network models
- Attacks & failures
- Framework for robustness
Cause: somewhere in Ohio
Cause: somewhere in Ohio
A few days earlier...

- “Alarm systems failed due to infection with Blaster Worm”

Introduction (1/2)

- Society is critically depending on complex networks
  - Internet
  - Transportation networks
  - Energy networks
  - Communication networks

- Severe consequences if networks are disrupted

- Robustness is defined as the extent to which the complex network is able to cope with perturbations imposed on it
Introduction (2/2)

• Examples of perturbations
  • Failures
    • Broken fibre cables
    • Malfunctioning switches
    • ...
  • Attacks
    • Denial-of-Service
    • Physical attack on Internet Exchange
    • ...

• Aim of this lecture: discuss examples of how graph-theoretic metrics can be used to quantify robustness

Motivation for virus spread in networks

• Computer viruses
  • security threat to Internet
  • annoyance
  • very costly
    • Code Red worm: several billion $$ in damage

• Why do we care?
  • Understanding the spread of a virus is the first step in preventing it
  • How fast do we need to disinfect nodes so that the virus dies quickly? Which nodes?
Applications of virus spread models

- Computer virus and worms modelling
- Epidemic algorithms
- Error propagation in networks
- Any self-replicating object on a dynamic network
- Emotions as infectious diseases in social networks

Simple SIS model (1)

- Homogeneous birth (infection) rate $\beta$ on all edges between infected and susceptible nodes
- Homogeneous death (curing) rate $\delta$ for infected nodes

$\tau = \beta/\delta$: effective spreading rate
Simple SIS model (2)

- Each node \( j \) can be in either of the two states:
  - “0”: healthy
  - “1”: infected
- Markov continuous time:
  - infection rate \( \beta \)
  - curing rate \( \delta \)
- Mathematically:
  - \( X_j \) is the state of node \( j \)
  - infinitesimal generator

\[
Q_j(t) = \begin{bmatrix}
-q_{1j} & q_{1j} \\
q_{2j} & -d_{2j}
\end{bmatrix}
\]

Simple SIS model (3)

- Nodes are interconnected in graph:
  \( Q_j(t) = \begin{bmatrix}
-q_{1j} & q_{1j} \\
q_{2j} & -d_{2j}
\end{bmatrix} \)

where the infection rate is due all infected neighbors of node \( j \):

\[
q_{ij}(t) = \beta \sum_{k=1}^{N} a_{jk} 1_{\{X_k(t) = 1\}}
\]

and where the adjacency matrix of the graph is

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{21} & a_{22} & \cdots & a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N1} & a_{N2} & \cdots & a_{NN}
\end{bmatrix}
\]
Simple SIS model (4)

- Markov theory requires that the infinitesimal generator is a matrix whose elements are NOT random variables
- However, this is not the case in our simple model:
  \[ q_{ij}(t) = \beta \sum_{k=1}^{N} a_{jk} 1_{\{x_i(t) = 1\}} \]
- By conditioning to each possible combination of infected states, we finally arrive to the exact Markov continuous SIS model
- **Drawback**: this exact model has \(2^N\) states, where \(N\) is the number of nodes in the network.
Simple SIS model (5): mean field

- The infinitesimal generator

\[
Q_j(t) = \begin{bmatrix} -q_{1j} & q_{1j} \\ \delta & -\delta \end{bmatrix}
\]

\[
q_{ij}(t) = \beta \sum_{k=1}^{N} a_{jk} 1\{X_k(t) = 1\}
\]

is replaced by its mean (the only approximation!)

\[
Q_j = \begin{bmatrix} -E[q_{1j}] & E[q_{1j}] \\ \delta & -\delta \end{bmatrix}
\]

\[
E[q_{ij}(t)] = \beta \sum_{k=1}^{N} a_{jk} \Pr\{\{X_k(t) = 1\}
\]

- Being able now to apply ordinary Markov theory, we arrive at our \(N\)-intertwined model for virus spread

\[
\frac{dv_i}{dt} = (1-v_i)\beta \sum_{k=1}^{N} a_{ik} v_k - \delta v_i
\]

\[
\frac{dv_2}{dt} = (1-v_2)\beta \sum_{k=1}^{N} a_{2k} v_k - \delta v_2
\]

\[
\vdots
\]

\[
\frac{dv_N}{dt} = (1-v_N)\beta \sum_{k=1}^{N} a_{Nk} v_k - \delta v_N
\]

where \(v_k(t) = \Pr[X_k(t) = 1]\)

N-intertwined virus spread model

- Non-linear matrix equation:

\[
\frac{dV(t)}{dt} = \beta A V(t) - \text{diag}(v_i(t))(\beta A V(t) + \delta u)
\]

where the vector \(u^T = [1 \ldots 1]\) and \(V^T = [v_1 \ v_2 \ldots v_N]\)

- Results:
  - Probability of infection \(v_k\) for each node \(k\) separately
  - Number of infected nodes in the steady state
  - Phase transition phenomena for any network (largest eigenvalue of the adjacency matrix \(A\))
  - Analytic computations feasible:
    - expansions of \(v_k\) as a function of the effective infection rate around the epidemic threshold and around infinity

Simulations

500 simulations

\[ K_{10.999} \quad \tau = \frac{1}{s} = 0.15 \]

\[ y_m(s) = \frac{(mn - s^2)}{m + n} \left( \frac{1}{s + m} + \frac{1}{s + n} \right) \]

Kephart-White model

Assume perfect homogeneity & symmetry: a graph of degree \( r \)

\[
\begin{align*}
\frac{dv_1}{dt} &= (1 - v_1) \beta \sum_{k=1}^{N} a_{ik} v_k - \delta v_1 \\
\frac{dv_2}{dt} &= (1 - v_2) \beta \sum_{k=1}^{N} a_{i2} v_k - \delta v_2 \\
&\vdots \\
\frac{dv_N}{dt} &= (1 - v_N) \beta \sum_{k=1}^{N} a_{iN} v_k - \delta v_N
\end{align*}
\]

\[ \frac{dv}{dt} = (1 - v) \beta rv - \delta v \]

steady-state

\[ v = 1 - \frac{1}{r \tau} \]

threshold

where \( \tau = \frac{\beta}{\delta} \)

\[ \tau \geq \tau_c = \frac{1}{r} \]

then \( v \geq 0 \)

What is so interesting about epidemics?

- Final epidemic state
- Rate of propagation
- Epidemic threshold

\[ \tau_c = \frac{1}{\lambda_1(A)} \]

\[ E[D] = \frac{2L}{N} \leq \lambda_1(A) \leq d_{\text{max}} \]

Affecting the epidemic threshold

- Degree-preserving rewiring
  - Changing the assortativity of the graph

- Removing links/nodes (optimal way is NP-complete)

- Quarantining: Removing inter-module links

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Framework for robustness

ResumeNet
A framework for network robustness: the R-model

EU project FP7 – 224619: http://www.resumenet.eu/
Network: topology + service(s)

- Topology (or network infrastructure):
  - graph $G$ with $N$ nodes and $L$ links
  - link weights
  - “hardware”

- Service:
  - more abstract and less clearly defined
  - uses the network infrastructure to transport items (e.g. email service, telephony, video, cars on roads, neurons in brain, etc.)
  - “software”

- Topology and service
  - own specifications
  - service is often designed independently of the topology
  - often more than 1 service on a same topology
**High-Level Goal:**
Express Network Resilience in a Number $R$

$R \in [0,1]$

- $R = 0$: absence of resilience
- $R = 1$: ideally resilient

**Simple framework**

- **network**
  - service
  - topology

- **Compute $R$-value**

- **Desired Graph**
  - yes
  - no

- **Modify Graph**

**Goals:**
1. define “$R$-value” that characterizes the level of robustness in *any* network
2. compute the $R$-value
3. robustness classes (understanding): which $R$ is desirable and what is $R_{\text{threshold}}$
**R-model**

\[ R = \sum_{k=1}^{m} s_k t_k = s^T t \quad (0 \leq R \leq 1) \]

- \( s \): the service vector with \( m \) components (interpreted as weights)
- \( t \): the topology vector where each component is a metric (e.g. average degree, clustering coefficient, algebraic connectivity, minimum degree, diameter/hopcount, betweenness, etc...)

- **Normalization:**
  - \( R = 0 \) (absence of network robustness)
  - \( R = 1 \) (perfect robustness)

- **Linear:**
  - simplest \( m \)-dim expression & geometric interpretation (vector)
  - expectation \( E[R] \) easy
  - no constraints on component values (else linear programming model)

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**Issues with R-model**

- **Dimension** \( m \): trade-off between accuracy and computational complexity (*not problematic, consensus*)

- **Orthogonality** of the metrics (*fundamental problem*)
  - each metric should ideally be a basis vector in \( m \)-dim space
  - almost all topology metrics are dependent
  - degree of dependence depends on the graph
  - **Solution**: no metrics, but matrices (adjacency \( A \), incidence \( B \), Laplacian \( Q \)) or spectra (graph theory)?

- **Normalization** of graphs: how to compare graphs with different number of nodes and links?

- unclear how to map a service onto a service vector (recall \( s_k \) is projection of \( s \) on \( k \)-th metric)
Which metrics to choose?

- Which metrics to choose is still an open question
  - Decomposition problem
  - Dependency problem
  - Normalization problem

User-Level Metrics

Topological Metrics

\[ R[k] \]

\[ R_{\text{threshold}} \]

\[ R \text{ as a function of “challenges”} \]

- resilience is related to the network’s capability to withstand perturbations from the outside during a given time interval
Understanding the Series of Events: Metric Envelopes

• Comparing resilience based on metric envelopes give a visual explanation of the network degradation process.

• Depending on the application domain a more bounded envelope might be preferable.

• The effect of various failure sources on the evaluated metric can be revealed.
Computational Approach to a Measuring Resilience

Case Study: A Wireless GEANT2

Show case for:
- Regional Challenges
- Fine-Grained, Intuitive Failures
W-GEANT2: Where are the weak points?

- Risk map indicates which areas are most vulnerable to challenges

Impact map visualizes the effect of a particular failure on the network as a whole

Let’s take a deeper look: What concretely would happen?

Outline

Introduction
Graph metrics
Spectrum
Network models
Attacks & failures
Framework for robustness

Books

Articles:  http://www.nas.ewi.tudelft.nl